

Chapter 12 Partial differential equations

$$\S \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \text{ with } u(x, y) = g(x, y) \text{ on } \partial\Omega,$$

Where

$$\Omega = \{(x, y) | a \leq x \leq b, c \leq y \leq d\}$$

$$\text{Set } x_0 = a, \quad x_{n+1} = b, \quad h = (b - a)/(n + 1), \quad x_i = x_0 + ih,$$

$$\text{Set } y_0 = c, \quad y_{m+1} = d, \quad k = (d - c)/(m + 1), \quad y_j = y_0 + jk,$$

$$u(x_i, y_j) = u_{i,j}, \quad f(x_i, y_j) = f_{i,j},$$

$$\frac{1}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + \frac{1}{k^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) = f_{i,j}, \quad \alpha = h^2/k^2$$

$$\alpha u_{i,j-1} + u_{i-1,j} - 2(1 + \alpha)u_{i,j} + u_{i+1,j} + \alpha u_{i,j+1} = h^2 f_{i,j}$$

Introduce the index (i, j) to be $i + n(j - 1)$

$$\text{Case 1 } j = 1: u_{i-1,1} - 2(1 + \alpha)u_{i,1} + u_{i+1,1} + \alpha u_{i,2} = h^2 f_{i,1} - \alpha u_{i,0}$$

$$\text{a. } i = 1: -2(1 + \alpha)u_{1,1} + u_{2,1} + \alpha u_{1,2} = h^2 f_{1,1} - \alpha u_{1,0} - u_{0,1} \triangleq F_{1,1}$$

$$-2(1 + \alpha)u_1 + u_2 + \alpha u_{1+n} = F_1$$

$$\text{b. } i = 2 \sim n - 1: u_{i-1} - 2(1 + \alpha)u_i + u_{i+1} + \alpha u_{i+n} = h^2 f_{i,1} - \alpha u_{i,0} \triangleq F_i$$

$$\text{c. } i = n: u_{n-1,1} - 2(1 + \alpha)u_{n,1} + \alpha u_{n,2} = h^2 f_{n,1} - \alpha u_{n,0} - u_{n+1,1} \triangleq F_{n,1}$$

$$u_{n-1} - 2(1 + \alpha)u_n + \alpha u_{2n} \triangleq F_n$$

$$\text{Case 2 } j = 2 \sim m - 1: \alpha u_{i,j-1} + u_{i-1,j} - 2(1 + \alpha)u_{i,j} + u_{i+1,j} + \alpha u_{i,j+1} = h^2 f_{i,j}$$

$$\text{a. } i = 1: \alpha u_{1,j-1} + u_{0,j} - 2(1 + \alpha)u_{1,j} + u_{2,j} + \alpha u_{1,j+1} = h^2 f_{1,j} - u_{0,j}$$

$$\alpha u_{1+n(j-2)} - 2(1 + \alpha)u_{1+n(j-1)} + u_{2+n(j-1)} + \alpha u_{1+nj} = h^2 f_{1,j} - u_{0,j} = F_{1+n(j-1)}$$

$$\text{b. } i = 2 \sim n - 1: \alpha u_{i+n(j-2)} + u_{i-1+n(j-1)} - 2(1 + \alpha)u_{i+n(j-1)} + u_{i+1+n(j-1)} + \alpha u_{i+nj} = h^2 f_{i+n(j-1)}$$

$$c. \quad i = n: \quad \alpha u_{n,j-1} + u_{n-1,j} - 2(1+\alpha)u_{n,j} + \alpha u_{n,j+1} = h^2 f_{n,j} - u_{n+1,j} \triangleq F_{n,j}$$

$$\alpha u_{n(j-1)} + u_{-1+nj} - 2(1+\alpha)u_{nj} + \alpha u_{n(j+1)} = F_{n(j-1)}$$

$$\text{Case 3 } j = m: \quad \alpha u_{i,m-1} + u_{i-1,m} - 2(1+\alpha)u_{i,m} + u_{i+1,m} = h^2 f_{i,m} - \alpha u_{i,m+1}$$

$$a. \quad i = 1: \quad \alpha u_{1,m-1} - 2(1+\alpha)u_{1,m} + u_{2,m} = h^2 f_{1,m} - \alpha u_{1,m+1} - u_{0,m} = F_{1,m}$$

$$\alpha u_{1+n(m-2)} - 2(1+\alpha)u_{1+n(m-1)} + u_{2+n(m-1)} = F_{1+n(m-1)}$$

$$b. \quad i = 2 \sim n-1: \quad \alpha u_{i,m-1} + u_{i-1,m} - 2(1+\alpha)u_{i,m} + u_{i+1,m} = h^2 f_{i,m} - \alpha u_{i,m+1} = F_{i,m}$$

$$\alpha u_{i+n(m-2)} + u_{i-1+n(m-1)} - 2(1+\alpha)u_{i+n(m-1)} + u_{i+1+n(m-1)} = F_{i+n(m-1)}$$

$$c. \quad i = n: \quad \alpha u_{n,m-1} + u_{n-1,m} - 2(1+\alpha)u_{n,m} = h^2 f_{n,m} - \alpha u_{n,m+1} - u_{n+1,m} \triangleq F_{n,m}$$

$$\alpha u_{n(m-1)} + u_{-1+nm} - 2(1+\alpha)u_{nm} = F_{nm}$$

The above equations can be assembled as the vector form

$$[A]_{nm \times nm} \{U\}_{nm} = \{F\}_{nm}.$$

Questions: How to implement a_{ij} ?

Consider $\beta u_k = F_l$, then $a_{lk} = \beta$.

§ Polar coordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = f(r, \theta), \quad a \leq r \leq b, \quad \theta_0 \leq \theta \leq \theta_{m+1}$$

with the boundary conditions $u(a, \theta)$, $u(b, \theta)$, $u(r, \theta_0)$ and $u(r, \theta_{m+1})$

Set $r_0 = a$, $r_{n+1} = b$, $h = (b-a)/(n+1)$, $k = (\theta_{m+1} - \theta_0)/(m+1)$,

$$r_i = r_0 + ih, \quad \theta_j = \theta_0 + jk, \quad u(r_i, \theta_j) = u_{i,j}, \quad f(r_i, \theta_j) = f_{i,j}$$

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{1}{r_i} \frac{u_{i+1,j} - u_{i-1,j}}{2h} + \frac{1}{r_i^2} \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = f_{i,j}$$

$$\alpha u_{i,j-1} + (r_i^2 - \frac{h}{2} r_i) u_{i-1,j} - 2(\alpha + r_i^2) u_{i,j} + (r_i^2 + \frac{h}{2} r_i) u_{i+1,j} + \alpha u_{i,j+1} = r_i^2 h^2 f_{i,j}$$

where $\alpha = (h/k)^2$.

Introduce the index (i, j) to be $i + n(j-1)$.

Case 1. $j=1$: $(r_i^2 - \frac{h}{2}r_i)u_{i-1,1} - 2(\alpha + r_i^2)u_{i,1} + (r_i^2 + \frac{h}{2}r_i)u_{i+1,1} + \alpha u_{i,2} = r_i^2 h^2 f_{i,1} - \alpha u_{i,0}$

a. $i=1$: $-2(\alpha + r_1^2)u_{1,1} + (r_1^2 + \frac{h}{2}r_1)u_{2,1} + \alpha u_{1,2} = r_1^2 h^2 f_{1,1} - \alpha u_{1,0} - (r_1^2 - \frac{h}{2}r_1)u_{0,1} \triangleq F_{1,1}$

$$-2(\alpha + r_1^2)u_1 + (r_1^2 + \frac{h}{2}r_1)u_2 + \alpha u_{1+n} = F_1$$

b. $2 \leq i \leq n-1$:

$$(r_i^2 - \frac{h}{2}r_i)u_{i-1,1} - 2(\alpha + r_i^2)u_{i,1} + (r_i^2 + \frac{h}{2}r_i)u_{i+1,1} + \alpha u_{i,2} = r_i^2 h^2 f_{i,1} - \alpha u_{i,0} \triangleq F_{i,1}$$

$$(r_i^2 - \frac{h}{2}r_i)u_{i-1} - 2(\alpha + r_i^2)u_i + (r_i^2 + \frac{h}{2}r_i)u_{i+1} + \alpha u_{i+n} = F_i$$

c. $i=n$:

$$(r_n^2 - \frac{h}{2}r_n)u_{n-1,1} - 2(\alpha + r_n^2)u_{n,1} + \alpha u_{n,2} = r_n^2 h^2 f_{n,1} - \alpha u_{n,0} - (r_n^2 + \frac{h}{2}r_n)u_{n+1,1} \triangleq F_{n,1}$$

$$(r_n^2 - \frac{h}{2}r_n)u_{n-1} - 2(\alpha + r_n^2)u_n + \alpha u_{2n} = F_n$$

Case 2 $2 \leq j \leq m-1$:

$$\alpha u_{i,j-1} + (r_i^2 - \frac{h}{2}r_i)u_{i-1,j} - 2(\alpha + r_i^2)u_{i,j} + (r_i^2 + \frac{h}{2}r_i)u_{i+1,j} + \alpha u_{i,j+1} = r_i^2 h^2 f_{i,j}$$

a. $i=1$:

$$\alpha u_{1,j-1} - 2(\alpha + r_1^2)u_{1,j} + (r_1^2 + \frac{h}{2}r_1)u_{2,j} + \alpha u_{1,j+1} = r_1^2 h^2 f_{1,j} - (r_1^2 - \frac{h}{2}r_1)u_{0,j} \triangleq F_{1,j}$$

$$\alpha u_{1+n(j-2)} - 2(\alpha + r_1^2)u_{1+n(j-1)} + (r_1^2 + \frac{h}{2}r_1)u_{2+n(j-1)} + \alpha u_{1+nj} = F_{1+n(j-1)}$$

b. $2 \leq i \leq n-1$:

$$\alpha u_{i+n(j-2)} + (r_i^2 - \frac{h}{2}r_i)u_{i-1+n(j-1)} - 2(\alpha + r_i^2)u_{i+n(j-1)} + (r_i^2 + \frac{h}{2}r_i)u_{i+1+n(j-1)}$$

$$+ \alpha u_{i+nj} = r_i^2 h^2 f_{i+n(j-1)}$$

c. $i=n$:

$$\alpha u_{n,j-1} + (r_n^2 - \frac{h}{2}r_n)u_{n-1,j} - 2(\alpha + r_n^2)u_{n,j} + \alpha u_{n,j+1} = r_n^2 h^2 f_{n,j} - (r_n^2 + \frac{h}{2}r_n)u_{n+1,j} \triangleq F_{n,j}$$

$$\alpha u_{n(j-1)} + (r_n^2 - \frac{h}{2}r_n)u_{-1+nj} - 2(\alpha + r_n^2)u_{nj} + \alpha u_{n(j+1)} = F_{nj}$$

Case 3 $j=m$:

$$\alpha u_{i,m-1} + (r_i^2 - \frac{h}{2}r_i)u_{i-1,m} - 2(\alpha + r_i^2)u_{i,m} + (r_i^2 + \frac{h}{2}r_i)u_{i+1,m} = r_i^2 h^2 f_{i,m} - \alpha u_{i,m+1}$$

a. $i=1$:

$$\alpha u_{1,m-1} - 2(\alpha + r_1^2)u_{1,m} + (r_1^2 + \frac{h}{2}r_1)u_{2,m} = r_1^2 h^2 f_{1,m} - \alpha u_{1,m+1} - (r_1^2 - \frac{h}{2}r_1)u_{0,m} \triangleq F_{1,m}$$

$$\alpha u_{1+n(m-2)} - 2(\alpha + r_1^2)u_{1+n(m-1)} + (r_1^2 + \frac{h}{2}r_1)u_{2+n(m-1)} = F_{1+n(m-1)}$$

b. $2 \leq i \leq n-1$

$$\alpha u_{i,m-1} + (r_i^2 - \frac{h}{2}r_i)u_{i-1,m} - 2(\alpha + r_i^2)u_{i,m} + (r_i^2 + \frac{h}{2}r_i)u_{i+1,m} = r_i^2 h^2 f_{i,m} - \alpha u_{i,m+1} \triangleq F_{i,m}$$

$$\alpha u_{i+n(m-2)} + (r_i^2 - \frac{h}{2}r_i)u_{i-1+n(m-1)} - 2(\alpha + r_i^2)u_{i+n(m-1)} + (r_i^2 + \frac{h}{2}r_i)u_{i+1+n(m-1)} = F_{i+n(m-1)}$$

c. $i=n$

$$\alpha u_{n,m-1} + (r_n^2 - \frac{h}{2}r_n)u_{n-1,m} - 2(\alpha + r_n^2)u_{n,m} = r_n^2 h^2 f_{n,m} - \alpha u_{n,m+1} - (r_n^2 + \frac{h}{2}r_n)u_{n+1,m} \triangleq F_{n,m}$$

$$\alpha u_{n(m-1)} + (r_n^2 - \frac{h}{2}r_n)u_{-1+nm} - 2(\alpha + r_n^2)u_{nm} = F_{nm}$$

The above equations can be assembled as the vector form

$$[A]_{nm \times nm} \{U\}_{nm} = \{F\}_{nm}.$$

Questions: How to implement a_{ij} ?Consider $\beta u_k = F_l$, then $a_{lk} = \beta$.

$$\S \quad \frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} + g(x, t), \quad 0 < x < l, \quad t > 0,$$

$$u(0, t) = q(t), \quad u(l, t) = r(t) \quad \text{for } t > 0, \quad \text{and } u(x, 0) = f(x) \quad \text{for } 0 \leq x \leq l.$$

$$\text{Set } x_0 = 0, \quad x_{n+1} = l, \quad h = l/(n+1), \quad x_i = x_0 + ih, \quad k = \Delta t, \quad t_j = jk, \quad u(x_i, t_j) = u_{i,j}$$

1. Forward-difference method

$$\frac{1}{k}(u_{i,j+1} - u_{i,j}) = \alpha^2 \frac{1}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + g_{i,j}$$

$$u_{i,j+1} = \lambda u_{i-1,j} + (1 - 2\lambda)u_{i,j} + \lambda u_{i+1,j} + kg_{i,j}, \quad \lambda = \alpha^2 \frac{k}{h^2}$$

$$u_{1,j+1} = \lambda u_{0,j} + (1 - 2\lambda)u_{1,j} + \lambda u_{2,j} + kg_{1,j} = (1 - 2\lambda)u_{1,j} + \lambda u_{2,j} + \lambda q_j + kg_{1,j},$$

$$u_{i,j+1} = \lambda u_{i-1,j} + (1 - 2\lambda)u_{i,j} + \lambda u_{i+1,j} + kg_{i,j}, \quad i = 2 \sim n-1,$$

$$u_{n,j+1} = \lambda u_{n-1,j} + (1-2\lambda)u_{n,j} + \lambda u_{n+1,j} + kg_{n,j} = \lambda u_{n-1,j} + (1-2\lambda)u_{n,j} + kg_{n,j} + \lambda r_j,$$

$$\begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_{j+1} = \begin{bmatrix} 1-2\lambda & \lambda & 0 & 0 \\ \lambda & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & \lambda \\ 0 & 0 & \lambda & 1-2\lambda \end{bmatrix} \begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_j + k \begin{Bmatrix} g_1 \\ \bullet \\ \bullet \\ g_n \end{Bmatrix}_j + \lambda \begin{Bmatrix} q \\ 0 \\ 0 \\ r \end{Bmatrix}_j, \quad \begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_0 = \begin{Bmatrix} f(x_1) \\ \bullet \\ \bullet \\ f(x_n) \end{Bmatrix}.$$

$$\text{Stability condition } \lambda < \frac{1}{2}$$

2. Backward-difference method

$$\frac{1}{k}(u_{i,j} - u_{i,j-1}) = \alpha^2 \frac{1}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + g_{i,j}$$

$$-\lambda u_{i-1,j} + (1+2\lambda)u_{i,j} - \lambda u_{i+1,j} - kg_{i,j} = u_{i,j-1}, \quad \lambda = \alpha^2 \frac{k}{h^2},$$

$$-\lambda u_{2,j} + (1+2\lambda)u_{1,j} - \lambda u_{0,j} - kg_{1,j} = u_{1,j-1},$$

$$-\lambda u_{i-1,j} + (1+2\lambda)u_{i,j} - \lambda u_{i+1,j} - kg_{i,j} = u_{i,j-1}, \quad i = 2 \sim n-1$$

$$-\lambda u_{n-1,j} + (1+2\lambda)u_{n,j} - \lambda u_{n+1,j} - kg_{n,j} = u_{n,j-1}$$

$$\begin{bmatrix} 1+2\lambda & -\lambda & 0 & 0 \\ -\lambda & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & -\lambda \\ 0 & 0 & -\lambda & 1+2\lambda \end{bmatrix} \begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_j - k \begin{Bmatrix} g_1 \\ \bullet \\ \bullet \\ g_n \end{Bmatrix}_j - \lambda \begin{Bmatrix} q \\ 0 \\ 0 \\ r \end{Bmatrix}_j = \begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_{j-1}, \quad \begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_0 = \begin{Bmatrix} f(x_1) \\ \bullet \\ \bullet \\ f(x_n) \end{Bmatrix}$$

3. Crank-Nicolson method

$$\frac{1}{k}(u_{i,j+1} - u_{i,j}) = \alpha^2 \frac{1}{2h^2}[(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + (u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1})] + g_{i,j}$$

or

$$-\frac{\lambda}{2}u_{i-1,j+1} + (1+\lambda)u_{i,j+1} - \frac{\lambda}{2}u_{i+1,j+1} = \frac{\lambda}{2}u_{i-1,j} + (1-\lambda)u_{i,j} + \frac{\lambda}{2}u_{i+1,j} + kg_{i,j}$$

$$(1+\lambda)u_{1,j+1} - \frac{\lambda}{2}u_{2,j+1} = (1-\lambda)u_{1,j} + \frac{\lambda}{2}u_{2,j} + kg_{1,j} + \frac{\lambda}{2}u_{0,j} + \frac{\lambda}{2}u_{0,j+1}$$

$$-\frac{\lambda}{2}u_{i-1,j+1} + (1+\lambda)u_{i,j+1} - \frac{\lambda}{2}u_{i+1,j+1} = \frac{\lambda}{2}u_{i-1,j} + (1-\lambda)u_{i,j} + \frac{\lambda}{2}u_{i+1,j} + kg_{i,j}, \quad i = 2 \sim n-1,$$

$$\begin{bmatrix} 1+\lambda & -0.5\lambda & 0 & 0 \\ -0.5\lambda & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & -0.5\lambda \\ 0 & 0 & -0.5\lambda & 1+\lambda \end{bmatrix} \{U\}^{(j+1)} = \begin{bmatrix} 1-\lambda & 0.5\lambda & 0 & 0 \\ 0.5\lambda & \bullet & \bullet & 0 \\ 0 & \bullet & \bullet & 0.5\lambda \\ 0 & 0 & 0.5\lambda & 1-\lambda \end{bmatrix} \{U\}^{(j)}$$

$$\S \quad \frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \quad t > 0, \quad u(0, t) = u(l, t) = 0, \quad u(x, 0) = f(x), \quad \frac{\partial u}{\partial t} = g(x)$$

$$\frac{1}{k^2} (u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) = \alpha^2 \frac{1}{h^2} (u_{i+1,j} - 2u_{i,j} + u_{i-1,j}),$$

$$u_{i,j+1} = \lambda^2 u_{i-1,j} + 2(1-\lambda^2)u_{i,j} + \lambda^2 u_{i+1,j} - u_{i,j-1}, \quad \lambda^2 = \alpha^2 \frac{k^2}{h^2},$$

$$u_{1,j+1} = 2(1-\lambda^2)u_{1,j} + \lambda^2 u_{2,j} - u_{1,j-1},$$

$$u_{i,j+1} = \lambda^2 u_{i-1,j} + 2(1-\lambda^2)u_{i,j} + \lambda^2 u_{i+1,j} - u_{i,j-1},$$

$$u_{n,j+1} = \lambda^2 u_{n-1,j} + 2(1-\lambda^2)u_{n,j} - u_{n,j-1}.$$

$$\begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_{j+1} = \begin{bmatrix} 2(1-\lambda^2) & \lambda^2 & 0 & 0 \\ \lambda^2 & 2(1-\lambda^2) & \lambda^2 & 0 \\ 0 & \lambda^2 & 2(1-\lambda^2) & \lambda^2 \\ 0 & 0 & \lambda^2 & 2(1-\lambda^2) \end{bmatrix} \begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_j - \begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_{j-1}$$

$$\text{Expand } u(x_i, t_1) = u(x_i, 0) + k \frac{\partial}{\partial t} u(x_i, 0) + \frac{1}{2} k^2 \frac{\partial^2}{\partial t^2} u(x_i, 0)$$

Or

$$u(x_i, t_1) = u(x_i, 0) + kg(x_i) + \frac{1}{2} k^2 \alpha^2 \frac{\partial^2}{\partial x^2} u(x_i, 0)$$

$$= f(x_i) + kg(x_i) + \frac{1}{2} k^2 \alpha^2 \frac{d^2}{dx^2} f(x_i)$$

$$= f(x_i) + kg(x_i) + \frac{1}{2} \frac{k^2 \alpha^2}{h^2} [f(x_{i+1}) - 2f(x_i) + f(x_{i-1}))],$$

$$= \frac{1}{2} \lambda^2 f(x_{i-1}) + (1-\lambda^2) f(x_i) + kg(x_i) + \frac{1}{2} \lambda^2 f(x_{i+1}),$$

So,

$$\begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_0 = \begin{Bmatrix} f(x_1) \\ \bullet \\ \bullet \\ f(x_n) \end{Bmatrix},$$

$$\begin{Bmatrix} u_1 \\ \bullet \\ \bullet \\ u_n \end{Bmatrix}_1 = \frac{1}{2}\lambda^2 \begin{Bmatrix} f(x_0) \\ \bullet \\ \bullet \\ f(x_{n-1}) \end{Bmatrix} + (1-\lambda^2) \begin{Bmatrix} f(x_1) \\ \bullet \\ \bullet \\ f(x_n) \end{Bmatrix} + k \begin{Bmatrix} g(x_1) \\ \bullet \\ \bullet \\ g(x_n) \end{Bmatrix} + \frac{1}{2}\lambda^2 \begin{Bmatrix} f(x_1) \\ \bullet \\ \bullet \\ f(x_n) \end{Bmatrix}$$

§ Finite Element Method

$$\frac{\partial}{\partial x}[p(x, y)\frac{\partial u}{\partial x}] + \frac{\partial}{\partial y}[q(x, y)\frac{\partial u}{\partial y}] + r(x, y)u(x, y) = f(x, y) \quad (1)$$

with the boundary conditions

$$u(x, y) = g(x, y) \quad \text{on } S_1 \quad \text{and}$$

$$p(x, y)\frac{\partial u}{\partial x}n_x + q(x, y)\frac{\partial u}{\partial y}n_y + g_1(x, y)u(x, y) = g_2(x, y) \quad \text{on } S_2 \quad (2)$$

Eq. (1) can be expressed as

$$\begin{aligned} \min I(u) = & \iint \left\{ \frac{1}{2} \left[p(x, y) \left(\frac{\partial u}{\partial x} \right)^2 + q(x, y) \left(\frac{\partial u}{\partial y} \right)^2 - r(x, y)u^2 \right] + f(x, y)u \right\} dx dy \\ & + \int_{S_2} \left\{ -g_2(x, y)u + \frac{1}{2}g_1(x, y)u^2 \right\} ds. \end{aligned} \quad (3)$$

Dividing the domain into n sub-domain and set the nodal variables of each sub-domain, then using the Lagrange's interpolation to set the function $u(x, y)$ in terms of the nodal variables as the form $u_i(x, y)$ in the local coordinates. Express the global variable $u(x, y)$ as the linear combination of $u_i(x, y)$ to yield

$$u(x, y) = \sum_{i=1} c_i u_i(x, y) \quad (4)$$

Substituting Eq. (4) into Eq. (3) yields

$$0 = \frac{\partial}{\partial c_i} \iint \left\{ \frac{1}{2} \sum_{i,j=1} c_i c_j \left[p \frac{\partial u_i}{\partial x} \frac{\partial u_j}{\partial x} + q \frac{\partial u_i}{\partial y} \frac{\partial u_j}{\partial y} - r u_i u_j \right] + \sum_{i=1} c_i f u_i \right\} dx dy$$

$$+ \frac{\partial}{\partial c_i} \int_{S_2} \left\{ - \sum_{i=1} c_i g_2 u_i + \frac{1}{2} \sum_{i,j=1} c_i c_j g_1 u_i u_j \right\} ds . \quad (5)$$

Or

$$[A]\{c\} = \{b\} , \quad (6)$$

where

$$a_{ij} = \iint \left[p \frac{\partial u_i}{\partial x} \frac{\partial u_j}{\partial x} + q \frac{\partial u_i}{\partial y} \frac{\partial u_j}{\partial y} - r u_i u_j \right] dx dy + \int_{S_2} g_1 u_i u_j dx dy ,$$

$$b_i = \iint f u_i dx dy - \int_{S_2} g_2 u_i ds .$$

Ex. Four nodes

$$u_1 = u(0,0) , \quad u_2 = u(l,0) , \quad u_3 = u(0,k) , \quad u_4 = u(l,k)$$

$$u(x,y) = u_1 \left(1 - \frac{x}{l}\right) \left(1 - \frac{y}{k}\right) + u_2 \left(\frac{x}{l}\right) \left(1 - \frac{y}{k}\right) + u_3 \left(1 - \frac{x}{l}\right) \left(\frac{y}{k}\right) + u_4 \left(\frac{x}{l}\right) \left(\frac{y}{k}\right)$$

Ex. Truss Element

$$\frac{d^2 u}{dx^2} = 0 \quad \text{with} \quad u(0) = u_1 , \quad u(l) = u_2$$

$$u^{(1)}(x) = (1 - x/l)u_1 + (x/l)u_2$$

Dividing the whole truss into n elements, then

$$u^{(i)}(x) = (1 - x/l)u_i + (x/l)u_{i+1}$$

$$= \{u_i \ u_{i+1}\} \{1 - x/l \ x/l\}^T = \{1 - x/l \ x/l\} \{u_i \ u_{i+1}\}^T$$

$$\text{The strain energy } S = \frac{1}{2} \int_0^L EA \left(\frac{du}{dx} \right)^2 dx = \frac{1}{2} \sum_{i=1}^n \int_0^l EA \left(\frac{du^{(i)}}{dx} \right)^2 dx$$

where E is the Young's modulus, A is the cross-sectional area.

$$S = \frac{1}{2} \sum_{i=1}^n \{u_i \ u_{i+1}\} \int_0^l EA \{1 - x/l \ x/l\}^T \{1 - x/l \ x/l\} dx \{u_i \ u_{i+1}\}^T$$

$$= \frac{1}{2} \sum_{i=1}^n \{u_i \ u_{i+1}\} \begin{bmatrix} k_{i \ i} & k_{i \ i+1} \\ k_{i+1 \ i} & k_{i+1 \ i+1} \end{bmatrix} \begin{Bmatrix} u_i \\ u_{i+1} \end{Bmatrix}$$

where

$$\begin{bmatrix} k_{i i} & k_{i i+1} \\ k_{i+1 i} & k_{i+1 i+1} \end{bmatrix} = \int_0^l EA \{1 - x/l \ x/l\}^T \{1 - x/l \ x/l\} dx$$

$$\begin{Bmatrix} \frac{\partial s}{\partial u_j} \\ \frac{\partial s}{\partial u_{j+1}} \end{Bmatrix} = \begin{bmatrix} k_{j j} & k_{j j+1} \\ k_{j+1 j} & k_{j+1 j+1} \end{bmatrix} \begin{Bmatrix} u_j \\ u_{j+1} \end{Bmatrix}$$

Combining all elements yields $[K]\{U\} = \{f\}$, $\{U\} = \{u_1 \ u_2 \ \dots \ u_{n+1}\}^T$

HW 12

1. Given the problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = xy, \quad 0 < x < \pi, \quad 0 < y < \pi/2$$

$$u(0, y) = \cos y, \quad u(\pi, y) = -\cos y, \quad 0 \leq y \leq \pi/2,$$

$$u(x, 0) = \cos x, \quad u(x, \pi/2) = 0, \quad 1 \leq y \leq 2$$

To calculate $u(x, y)$ by using $h = k = 0.1\pi$.

2. Given the problem

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} = \frac{1}{4K} \frac{\partial T}{\partial t}, \quad \frac{1}{2} \leq r \leq 1, \quad 0 \leq t,$$

$$T(1, t) = 100 + 40t, \quad 0 \leq t \leq 10; \quad \frac{\partial T}{\partial r} + 3T = 0 \quad \text{at} \quad r = \frac{1}{2}$$

$$T(r, 0) = 200(r - 0.5), \quad 0.5 \leq r \leq 1,$$

and use $\Delta t = 0.5$, $\Delta r = 0.1$, and $K = 0.1$ to calculate $T(r, t)$

By (a) the forward-difference method

(b) the backward-difference method

© the Crank-Nicolson algorithm.

3. Given the problem

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} = 0, \quad \frac{1}{2} \leq r \leq 1, \quad 0 \leq t \leq \pi/3,$$

$$T(r, 0) = 0, \quad T(r, \pi/3) = 0, \quad T(1/2, \theta) = 50, \quad T(1, \theta) = 100.$$

4. Given the problem

$$\frac{\partial^2 p}{\partial t^2} = \frac{\partial^2 p}{\partial x^2}, \quad 0 \leq x \leq 1, \quad 0 \leq t$$

$$p(0,t)=1, \quad p(1,t)=2, \quad p(x,0)=\cos(2\pi x), \quad \frac{\partial p}{\partial t}(x,0)=2\pi \sin(2\pi x), \quad 0 \leq x \leq 1$$

To calculate p by using $\Delta x = \Delta t = 0.1$.